TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

ANNOTATING LATTICE ORBIFOLDS WITH MINIMAL ACTING AUTOMORPHISMS

Tobias Schlemmer

Málaga, 2012-10-14

Outline

History

1991 Monika Zickwolff: First Order Logic in FCA

2009 Daniel Borchmann/Bernhard Ganter: Context Orbifolds

Idea

Compression of binary relations to

- reduce storage need,
- get new insight into the structure, and
- reduce processing time

Set matching in Music

Notions (simplified)

Chroma system: $C := \mathbb{Z}_{12}$

Harmony: $H \subseteq C$

Harmonic Form: $F(H) := \{H + i \mid i \in \mathbb{Z}_{12}\}$

Question

Is the form of a harmony matched by a set $\mathfrak{S}_H$ of harmonic forms?

Notions (simplified)

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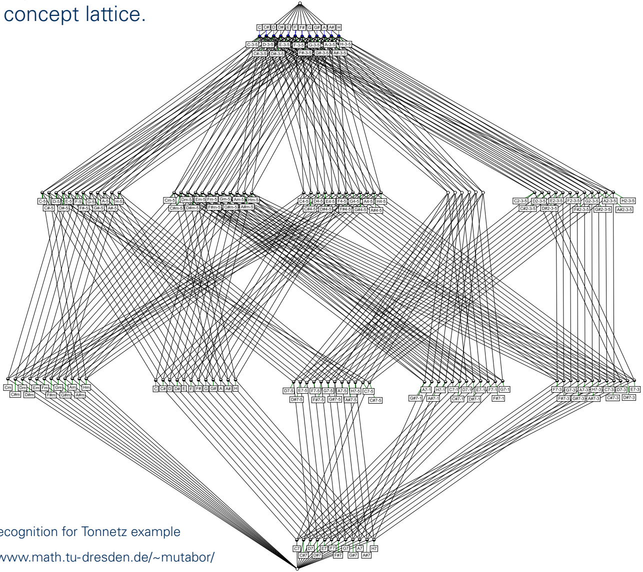
Question

Is the form of a harmony matched by a set $\mathfrak{S}_H$ of harmonic forms?

Transpositions preserve the structure of…

- …the set of harmonies of a harmonic form,
- …the chroma system,
- …the context formed by harmonies, chomas and the element relationship, and…

. . . its concept lattice.



Pattern recognition for Tonnetz example

at http://www.math.tu-dresden.de/~mutabor/

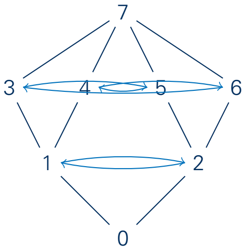
Topic

Definition (Automorphism)

- Lattice $(L, \leq)$
- Automorphism φ: isomorphism $\varphi : L \to L$, i. e.
  $\forall x, y \in L : x \leq y \Leftrightarrow x^\varphi \leq y^\varphi$.
- Automorphism Group $\mathfrak{Aut}(L, \leq) := \{\varphi : L \to L \mid \varphi \text{ is an automorphism}\}$

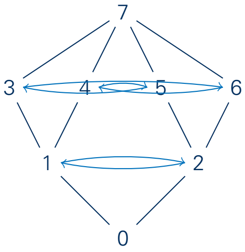
Folding group

$\mathbb{G} \leq \mathfrak{Aut}(L, \leq)$



Automorphisms

$(12)(3546)$



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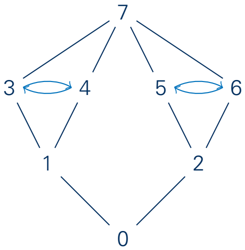

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(34), (56),
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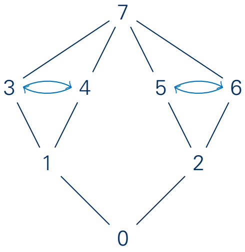
Example



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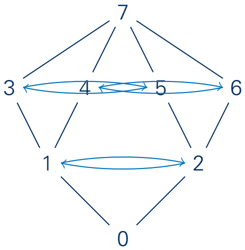

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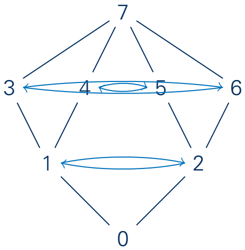

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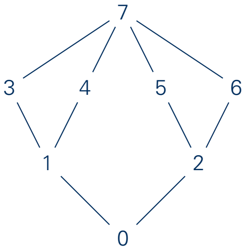

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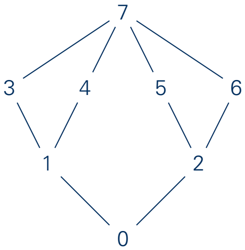

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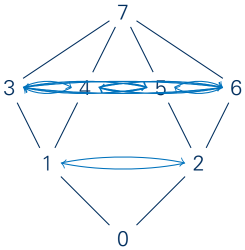

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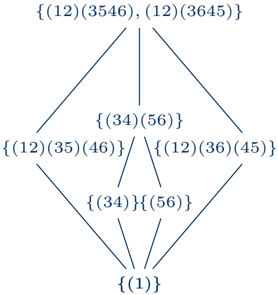
Orbits

Minimal sets that are invariant under group action.
$M \backslash\backslash \mathbb{G} = \{\{0\}, \{1,2\}, \{3,4,5,6\}, \{7\}\}$

2 Orbifolds

Preorder of Automorphisms



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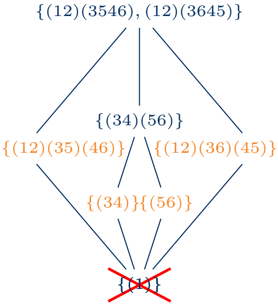
Preorder of Automorphisms

- $\varphi \sqsubseteq \psi :\Leftrightarrow \forall x \in L : x^{\langle \varphi \rangle} \subseteq x^{\langle \psi \rangle}$
- Minimal Acting Automorphism: $\varphi \in U \subseteq G$: $\forall \psi \in U : \psi \sqsubseteq \varphi \Rightarrow \varphi \sqsubseteq \psi$

Preorder of Automorphisms



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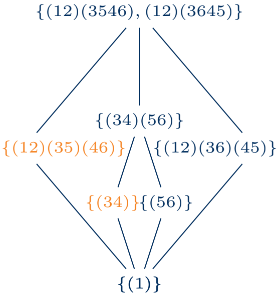

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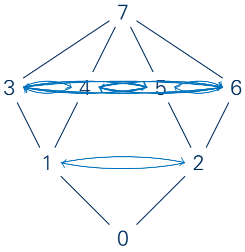

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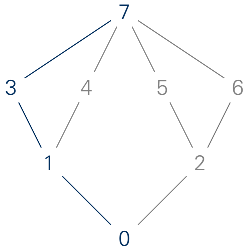
Transversal

A set consisting of exactly 1 representant of each orbit
$T := \{0, 1, 3, 7\}$

Annotation

$\lambda : T \times T \to G : (x, y) \mapsto \{\varphi \in G \mid x \leq y^{\varphi}\}$
$\lambda(0, 1) = \lambda(0, 3) = \lambda(0, 7) = \lambda(3, 7) = G,$
$\lambda(1, 3) = \lambda(1, 7) = \{(1), (34), (56), (34)(56)\}$

Example: Annotations



Full

7

$\Big|$ G

3

$\Big|$ $(1), (34), (56), (34)(56)$

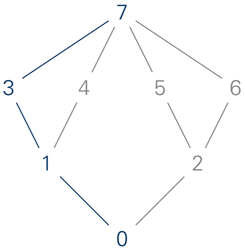
1

$\Big|$ G

0

$G = \{(34), (56), (34)(56), (12)(35)(46), (12)(36)(45), (12)(3546), (12)(3645)\}$

Example: Annotations



Full

7

| G

3

| (1), (34), (56), (34)(56)

1

| G

0

$G = \{(34), (56), (34)(56), (12)(35)(46), (12)(36)(45), (12)(3546), (12)(3645)\}$

Abridged

7 G

(1) |

3 (1), (56)

(1) |

1 (1), (34), (56), (34)(56)

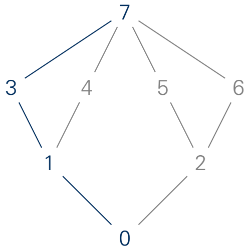
(1) |

0 G

$G = \{(34), (56), (34)(56), (12)(35)(46), (12)(36)(45), (12)(3546), (12)(3645)\}$

2 Orbifolds

Example: Annotations



Full

$$\begin{tikzcd} 7 \arrow[d, dash, "G"] \\ 3 \arrow[d, dash, "{(1),(34),(56),(34)(56)}"] \\ 1 \arrow[d, dash, "G"] \\ 0 \end{tikzcd}$$

$G = \{(34), (56), (34)(56), (12)(35)(46), (12)(36)(45), (12)(3546), (12)(3645)\}$

Abridged

$$\begin{tikzcd} 7 \;\; G \arrow[d, dash, "(1)"'] \\ 3 \;\; (1),(56) \arrow[d, dash, "(1)"'] \\ 1 \;\; (1),(34),(56),(34)(56) \arrow[d, dash, "(1)"'] \\ 0 \;\; G \end{tikzcd}$$

$G = \{(34), (56), (34)(56), (12)(35)(46), (12)(36)(45), (12)(3546), (12)(3645)\}$

Hierarchical

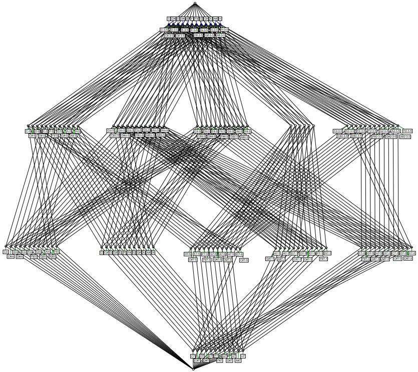
$$\begin{tikzcd} 7 \\ 3 \arrow[u, "(1)"] \\ 1 \arrow[u, "{(1),(43)}"] \\ 0 \arrow[u, "{(1),(21)(53)(64)}"] \end{tikzcd}$$

$<(21)(53)(64), (43)> \subseteq G$

Folding the Automaton

Data structures

- For each transition $(A, B) \to (C, D)$ store automorphism φ and character c such that $(B \cup \{c\})^\varphi \subseteq D$
- For current state store pointer to the concept orbit representative and automorphism that must be applied to the input



Topic

Conclusion

Advantages

- Less information stored
- Usable in applications

Disadvantages

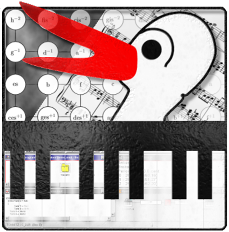
- G may not be reconstructable
- Local view on elements $x, y \in L$ needs to consider the Union $[0, x] \cup [0, y]$.

Further improvements

- “nice” theory (replace technical definitions and limitations)
- Impact of modifications of the context

Questions? Otázky? Вопросы? Fragen?

Thank you for your attention!



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Construction of the Folded Automaton

Construction of Folded Automaton

Store necessary automorphisms in transitions and current state

1. For each harmonic form select one representative
2. Form the power set of each representative
3. Find each subset (ordered by increasing size) in the intermediate lattice
   - If not found: Add the subset to the lattice
4. Result: Set recognition automaton

A Concept Lattice for Set Matching

System of Sets

- Base set S
- Set of subsets $\mathfrak{S} \subseteq \mathfrak{P}(S)$

Formal Context

- $\mathbb{K}(G, S, \ni)$
- $G = \bigcup\{\mathfrak{P}(A) \mid A \in \mathfrak{S}\}$

Concept Lattice

- $\mathfrak{B}(G, S, \ni)$

Simple set matching

Set matching automaton

- Start: $(G, \emptyset)$
- Alphabet: $S \cup \{\int\}$
- State: Concept $(A, B) \in \mathfrak{B}(G, S, \ni)$
- Transition at Element $s \neq \int$: $((A, B), s) \mapsto (A, B) \wedge \mu s$
- End at Element $\int$: "accept" if $B \in \mathfrak{S}$, "reject" otherwise.

Folding the Automaton

Simple approach

- Consider the harmony as a sequence of numbers
- Expand $\mathfrak{S}_H$ into a set of harmonies
- Drawback: Lattice may be much larger than necessary

Data structures

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